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A Class of Renormings of l^p with the Fixed Point Property

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Abstract: One of the open problems in metric fixed point theory is: Does every equivalent renorming of $l^p, 1 < p < \infty,$ have the fixed point property? In this paper, we give some classes of renormings of $l^p, 1 < p < \infty,$ with the fixed point property. It is known that $(l^p, \|.\|_p), 1 < p < \infty,$ has the fpp.

Keywords: Fixed point property, Nonexpansive map, Reflexive Banach space.

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1. Introduction

The most crucial result in metric fixed point theory is Banach contraction principle, given by S. Banach in 1922. In 1965, the discovery of a key fixed point theorem for nonexpansive mappings gave the basis for the development of consequent theory in the area of metric fixed point theory.

Definition 1.1. Let K be a closed bounded convex subset of a Banach space X. A mapping $T: K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K$.

We say that K has the fixed point property (fpp) if every nonexpansive mapping on K has a fixed point in K. Also, the Banach space X has the fpp if every closed bounded convex subset of X has the fpp. Generally, a Banach space X may not have the fpp. The following example shows that c_0 does not have the fpp.

Example 1.2. Let K be the closed unit ball in c_0 with the sup norm. Define, $T: K \to K$ as $T(x_1, x_2, \dots) = (1, x_1, x_2, \dots), \forall x = (x_1, x_2, \dots) \in K$. Then T is a nonexpansive map on K without any fixed point in K.

Although, a closed bounded convex subset K of a Banach space may fail to have the fpp but there always exists a sequence in K called the approximate fixed point sequence (afps) of the nonexpansive map T on K. The following theorem ensures the existence of such a sequence.

Theorem 1.3. [10] Let K be a closed bounded convex subset of a Banach space X and $T: K \to K$ be a nonexpansive map. Then T has an afps in K i.e. there exists a sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$.

In 1965, Browder and Gohde [10] proved that every uniformly convex Banach space has the fpp. In the same year Kirk [13] established that every reflexive Banach space with normal structure has the fpp. Later, Karlovitz [11] showed that normal structure is not a necessary condition for the existence of the fpp for a Banach space.

Definition 1.4. [3,10] Let C be a convex subset of a Banach space X. The set C is said to have normal structure if for every closed bounded convex subset K of C with $diam(K) > 0$, there exists a point $x \in K$ such that $r_x(K) = \sup\{\|x - y\| : y \in K\} < diam(K)$, the diameter of K. The Banach space X is said to have normal structure if every convex subset of X has normal structure.

Originally, the geometric property normal structure was introduced by Brodskii and Milman [3] in 1948 to study the existence of common fixed points for isometries. It is known that every uniformly convex Banach space has normal structure [10] and hence l^p has normal structure for $1 < p < \infty$. In metric fixed point theory, a long time open question is: "Does every reflexive Banach space have the fpp?"

In 1997, Lennard and Dowling [6] established that a subspace of $L^1([0,1])$ has the fpp if and only if it is reflexive. In 2009, Benavides [5] showed that in every reflexive Banach space there exists an equivalent renorm such that it has the fpp with respect to this new renorming.

A specific case of the above question is: "Does every superreflexive Banach space have the fpp?"Although, Maurey [1] proved that superreflexive spaces have the fpp for isometries, the question for general nonexpansive mappings remains unsolved.

Definition 1.5. [1] A Banach space X is called superreflexive if and only if it is uniformly convex with respect to some equivalent renorming.

Even more limiting problem is: "Does every equivalent renormings of $l^p, 1 < p < \infty,$ has the fpp?" Note that $l^p, 1 < p < \infty,$ spaces are superreflexive. Inspite of being a numerous numbers of studies [1,2,8,10,12,14,15,16] the problem remains unsolved. In 1985, Lin [15] proved that $\left(l^p, |.|_\beta\right), 1 < p < \infty,$ where $|x|_\beta = \max\{\|x\|_p, \beta\|x\|_\infty\}$, have the fpp for all $\beta > 0$, using unconditional bases concept. Recently, Piatek [18] gave some examples new equivalent renormings of $l^p, 1 < p < \infty,$ with the fpp.

In this paper, we prove that the renorming $\left(l^p, |.|_\beta\right), 1 < p < \infty,$ where $|x|_\beta = \max\{\|x\|_p, \beta q(x)\}$, ($q$ is a seminorm on l^p and $\beta > 0$) has the fpp, for all $\beta > 0$, if there is a non negative integer N satisfying the following condition:

Let $x \in l^p$ and q(x) > a, for some a > 0. Then there exist $j, m \in \mathbb{N}$ and non negative integers $n_1 < n_2 < \cdots < n_m$ such that

$$q(x) \geq |x(j + n_1) + \cdots + x(j + n_m)| > a,$$

$$q(x - z) \geq |(x - z)(j + n_1) + \cdots + (x - z)(j + n_m)| > a,$$

for all $z \in l^p$, where $n_m - n_1 \leq N$.

2. Main Results

Let K be a nonempty closed bounded convex subset of a Banach space X and T be a nonexpansive mapping on K. A nonempty closed bounded convex subset C of K is said to be minimal invariant under T if C is invariant under T and no proper nonempty closed bounded convex subset of C is invariant under T. The next theorem guarantees the existence for a minimal invariant set.

Theorem 2.1. [10] *Let K be a closed bounded convex subset of a reflexive Banach space X. Let T be a nonexpansive self mapping on K. Then there exists a closed bounded convex subset C of K which is minimal invariant under T.*

The following theorem yields a characterization for normal structure.

Theorem 2.2. [10] *A nonempty bounded convex subset K of a Banach space X has normal structure if and only if it does not contain a non constant sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \dots, x_n\}) = diam\{x_1, x_2, \dots, x_n, \dots\}.$$

The succeeding theorem is famous as Goebel Karlovitz lemma.

Theorem 2.3. [11] *Let X be a reflexive Banach space. Let K be a nonempty weakly compact convex subset of X and $T: K \to K$ be a nonexpansive mapping. Suppose K is minimal invariant under T and $\{x_n\}$ is an afps of T in K. Then $\lim_{n\to\infty} \| x_n - x\| = diam(K)$, for all $x \in K$.*

An important outcome in the way of fpp is given by Maurey [16] which is the following theorem.

Theorem 2.4. [4,16] *Let K be a nonempty closed bounded convex subset of a Banach space X. Suppose $T: K \to K$ is a nonexpansive map and $\{x_n\}, \{y_n\}$ are afps of T. Then there exists an afps $\{z_n\}$ of T such that*

$$\lim_{n\to\infty} \| x_n - z_n\| \leq \frac{diam(K)}{2}$$
$$\lim_{n\to\infty} \| y_n - z_n\| \leq \frac{diam(K)}{2}.$$

Consider $l^p, 1 < p < \infty$, with the norm $|x|_\beta = max\{\|x\|_p, \beta q(x)\}$, where $\beta > 0$ and q is a seminorm on l^p. This norm is an equivalent renorming of $\|x\|_p$ if and only if there exists $M > 0$ such that $q(x) \leq M\|x\|_p$, for all $x \in l^p$. Now forth, throughout we use $|x|_\beta$ to refer the above renorming of l^p.

In [9] it is proved that $\left(l^p, |.|_\beta\right), 1 < p < \infty$, has normal structure and hence the fpp if q fulfills certain assumption (Since the Schauder standard basis $\{e_n\} = \{(x(1), \dots, x(n), \dots): x(n) = 1, x(i) = 0, i \neq n\}$ of $l^p, 1 < p < \infty$, is unconditional with unconditional basis constant 1). Here we give a condition on the seminorm q such that $\left(l^p, |.|_\beta\right), 1 < p < \infty$, has the fpp, for all $\beta > 0$, but theses spaces may not posses normal structure.

Theorem 2.5. [7,8] *Let K be a closed bounded convex subset of $l^p, 1 < p < \infty$. Suppose $\{x_n\}$ is a sequence in K such that $|x_n - x|_\beta \to d = diam_\beta(K)$, for all $x \in K$. Then $q(x_n - x) \to d/\beta$, for all $x \in K$.*

The following lemma can be deducted for the Banach space $l^p, 1 < p < \infty$ from Lemma 27 in [8] by pursuing the same proof steps.

Lemma 2.6. *Let N be a non negative integer and $\{x_n\}$ be a sequence in $l^p, 1 < p < \infty$, such that $x_n \to^w 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{u_k\}$ such that*

(i) $\left|x_{n_k} - u_k\right|_\beta \to 0$

(ii) $\|u_k\|_p \leq \left\|x_{n_k}\right\|_p$, for all k.

(iii) There exist positive integers $N_0, N_1, \dots, N_k, \dots$ such that $N_{k-1} + N < N_k, k = 1,2, \cdots$ and $supp(u_k) \subset (N_{k-1}, N_k]$ where $supp(x) = \{n \in \mathbb{N}: x(n) \neq 0\}$ and $(N_{k-1}, N_k] = \{j \in \mathbb{N}: N_{k-1} < j \leq N_k\}$.

Theorem 2.7. *The renorming $\left(l^p, |.|_\beta\right), 1 < p < \infty$, has the fpp, for all $\beta > 0$, if there is a non negative integer N satisfying the following condition:*

Let $x \in l^p$ and q(x) > a, for some a > 0. Then there exist $j, m \in \mathbb{N}$ and non negative integers $n_1 < n_2 < \cdots < n_m$ such that

$$q(x) \geq |x(j + n_1) + \cdots + x(j + n_m)| > a,$$

$$q(x - z) \geq |(x - z)(j + n_1) + \cdots + (x - z)(j + n_m)| > a,$$

for all $z \in l^p$, where $n_m - n_1 \leq N$.

Proof. Suppose $\left(l^P, |.|_\beta\right)$ does not have the fpp. Then there exists a closed bounded convex set $K$and a nonexpansive map $T: K \to K$ which has no fixed point in K. We can assume that K is minimal invariant and $diam_\beta(K) = 1$ by a division.

Let $\{x_n\}$ be an afps of T in K which exists by Theorem 1.3. Since $l^p, 1 < p < \infty$, is reflexive, after passing through a subsequence and applying a translation we can also take that $0 \in K$ and $\{x_n\}$ converges weakly to 0.

By Lemma 2.6, there is a sequence $\{u_k\}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\left|x_{n_k} - u_k|_\beta \to 0\right.$ and $supp(u_k) \subset (N_{k-1}, N_k]$, where $N_{k-1} + N < N_k, k = 1,2, \dots$

Now, $\{x_{n_k}\}$ and $\{x_{n_{k+2}}\}$ are subsequence of $\{x_n\}$ and hence both are afps of T. Then by Theorem 2.4, there exists an afps $\{z_k\}$ of T such that

$$lim_{n\to\infty}\left|x_{n_k} - z_k\right|_\beta \leq \frac{1}{2},\ lim_{n\to\infty}\left|x_{n_{k+2}} - z_k\right|_\beta \leq \frac{1}{2}$$

Therefore,

$$lim_{n\to\infty}|u_k - z_k|_\beta \leq \frac{1}{2},\ lim_{n\to\infty}|u_{k+2} - z_k|_\beta \leq \frac{1}{2}.$$

Now by Goebel Karlovitz Lemma (Theorem 2.3), $|z_k - x|_\beta \to 1$, for all $x \in K$.

Then from Theorem 2.5, $q(z_k - x) \to \frac{1}{\beta}$, for all $x \in K$. So, $q(z_k) \to 1/\beta$ (since $0 \in K$).

Let $0 < \epsilon < \frac{1}{4}$. Choose, $k_0 \in \mathbb{N}$ such that

$$q(z_{k_0}) > \frac{(1-\epsilon)}{\beta}$$

$$\left|u_{k_0} - z_{k_0}\right|_\beta < \frac{1}{2} + \epsilon \Rightarrow q(u_{k_0} - z_{k_0}) < \frac{\left(\frac{1}{2} + \epsilon\right)}{\beta}$$

$$\left|u_{k_0+2} - z_{k_0}\right|_\beta < \frac{1}{2} + \epsilon \Rightarrow q(u_{k_0+2} - z_{k_0}) < \frac{\left(\frac{1}{2} + \epsilon\right)}{\beta}$$

So, by the assumption there exist $j, m \in \mathbb{N}$ and non negative integers $n_1 < n_2 < \cdots < n_m$ such that

$$q(z_{k_0}) \geq \left|z_{k_0}(j+n_1) + \cdots + z_{k_0}(j+n_m)\right| > \frac{(1-\epsilon)}{\beta},$$

$$q(z_{k_0} - z) \geq \left|(z_{k_0} - z)(j+n_1) + \cdots + (z_{k_0} - x)(j+n_m)\right|,$$

for all $z \in l^p$, where $n_m - n_1 \leq N$.

Now,

$$\left|u_{k_0}(j+n_1) + \cdots + u_{k_0}(j+n_m)\right| \geq \left|z_{k_0}(j+n_1) + \cdots + z_{k_0}(j+n_m)\right|$$

$$-\left|(z_{k_0} - u_{k_0})(j+n_1) + \cdots + (z_{k_0} - u_{k_0})(j+n_m)\right|$$

$$> \frac{(1-\epsilon)}{\beta} - q(u_{k_0} - z_{k_0}) > \frac{(1-\epsilon)}{\beta} - \frac{\left(\frac{1}{2} + \epsilon\right)}{\beta} > 0$$

Similarly, $\left|u_{k_0+2}(j+n_1) + \cdots + u_{k_0+2}(j+n_m)\right| > 0$.

Since $\left|u_{k_0}(j+n_1) + \cdots + u_{k_0}(j+n_m)\right| > 0$, there exists n_t such that $u_{k_0}(j+n_t) > 0$.

So, $j + n_t \in (N_{k_0-1}, N_{k_0}]$.

Also since $|u_{k_0+2}(j+n_1)+\cdots+u_{k_0+2}(j+n_m)|>0$, there exists n_s such that $u_{k_0+2}(j+n_s)>0$. So, $j+n_s\in(N_{k_0+1},N_{k_0+2}]$.

Now, $N_{k_0}+N<N_{k_0+1}$. Thus, we have $|n_t-n_s|>N$, which is a contradiction.

Example 2.8. Consider $l^p, 1<p<\infty$,with the renorming $\|x\|=max\{\|x\|_p,\beta\|x\|_\infty\},\beta>0$. It is known that l^p with this renorming has normal structure if and only if $\beta<2^{1/p}$[7].

It is also known that it has the fpp for all $\beta>0$ [1,7,15]. Here, we give a proof for the fpp of l^p with this renorming using Theorem 2.7.

Take $q(x)=\|x\|_\infty$. Let $x\in l^p$ and $q(x)>a$, for some $a>0$. Then there exists a positive integer $j$such that

$$q(x)\geq|x(j)|>a,$$

$$q(x-z)=\|x-z\|_\infty\geq|(x-z)(j)|,\forall z\in l^p.$$

Now by taking $N=0,m=1,n_1=0$ in Theorem 2.7, the result follows.

Example 2.9. Consider $X=l^p,1<p<\infty$, with the renorming

$\|x\|=max\{\|x\|_p,\beta\sup_n|x(n+1)+\cdots+x(n+i)|\}$, where $i\geq1$ is an integer and $\beta>0$.

Let $q(x)=\sup_n|x(n+1)+\cdots+x(n+i)|$. Then $q(x)\leq i\|x\|_\infty\leq i\|x\|_p$.

Now, when $\beta\geq2^{1/p}$, X does not have normal structure.

To see this take the sequence $\{e_n\}$. Then $\|e_n-e_m\|=\beta, n\neq m\Rightarrow diam(\{e_n\})=\beta$.

Also, $lim_{n\to\infty}dist\,(e_{n+1},co\{e_1,\cdots,e_n\})=\beta$.

So, by Theorem 2.2, X does not have normal structure when $\beta\geq2^{1/p}$.

However, we do not know for the case $\beta<2^{1/p}$ but we can show that these spaces have the fpp for all $\beta>0$.

Let $x\in X$ and $q(x)>a$, for some $a>0$. Then there exists a positive integer j such that

$$q(x)\geq|x(j+1)+\cdots+x(j+i)|>a,$$

$$q(x-z)\geq|(x-z)(j+1)+\cdots+(x-z)(j+i)|,\forall z\in l^p.$$

So, in Theorem 2.7, choosing $N=(i-1),m=i,n_1=1,\ldots,n_i=i$, we get the result.

Example 2.10. Let $X=l^p,1<p<\infty$, with the renorming

$$\|x\|=\max\{\|x\|_p,a\sup_{n\geq2}|x(1)+x(n)+x(n+1)|,b\|x\|_\infty\},\text{ where }a>0,b\geq0.$$

Let $\beta=\max\{a,b\}$.

Let $q(x) = \max\{\frac{a}{\beta}\sup_{n\geq 2}|x(1) + x(n) + x(n+1)|, \frac{b}{\beta}\|x\|_\infty\}$.

Now, when $\max\{a, b\} \geq 2^{1/p}$, the sequence $\{e_n\}_{n\geq 2}$ satisfies

$$\lim_{n\to\infty} dist\,(e_{n+1}, co\{e_2, \dots, e_n\}) = max\{a, b\} = diam\{e_n\}.$$

Hence by Theorem 2.2, X does not have normal structure when $\max\{a, b\} \geq 2^{1/p}$.

Now let us prove that X has the fpp, for all $\beta > 0$.

Suppose X does not have the fpp. Then as in Theorem 2.7, there exist sequences $\{u_k\}$ and $\{z_k\}$ such that $supp(u_k) \subset (N_{k-1}, N_k]$, where $N_{k-1} + 1 < N_k$ and

$$q(z_k) \to \frac{1}{\beta},$$

$$\lim_{n\to\infty}|u_k - z_k|_\beta \leq \frac{1}{2}, \lim_{n\to\infty}|u_{k+2} - z_k|_\beta \leq \frac{1}{2}.$$

Let $0 < \epsilon < \frac{1}{4}$. Choose $k_0 \in \mathbb{N}$ such that $N_{k_0-1} > 1$ and

$$q(z_{k_0}) > \frac{1-\epsilon}{\beta}$$

$$|u_{k_0} - z_{k_0}|_\beta < \frac{1}{2} + \epsilon \Rightarrow q(u_{k_0} - z_{k_0}) < \frac{\frac{1}{2}+\epsilon}{\beta},$$

$$|u_{k_0+2} - z_{k_0}|_\beta < \frac{1}{2} + \epsilon \Rightarrow q(u_{k_0+2} - z_{k_0}) < \frac{\frac{1}{2}+\epsilon}{\beta}.$$

Now

$$q(z_{k_0}) = \max\{\frac{a}{\beta}\sup_{n\geq 2}|z_{k_0}(1) + z_{k_0}(n) + z_{k_0}(n+1)|, \frac{b}{\beta}\|z_{k_0}\|_\infty\}.$$

Then there exists a positive integer j such that

$$\frac{a}{\beta}|z_{k_0}(1) + z_{k_0}(j) + z_{k_0}(j+1)| > \frac{1-\epsilon}{\beta}$$

$$\text{or, } \frac{b}{\beta}|z_{k_0}(j)| > \frac{1-\epsilon}{\beta}.$$

Suppose, $\frac{a}{\beta}|z_{k_0}(1) + z_{k_0}(j) + z_{k_0}(j+1)| > \frac{1-\epsilon}{\beta}$. Then

$$\frac{a}{\beta}|u_{k_0}(1) + u_{k_0}(j) + u_{k_0}(j+1)| \geq \frac{a}{\beta}|z_{k_0}(1) + z_{k_0}(j) + z_{k_0}(j+1)|$$

$$-\frac{a}{\beta}|(z_{k_0} - u_{k_0})(1) + (z_{k_0} - u_{k_0})(j) + (z_{k_0} - u_{k_0})(j+1)|$$

$$> \frac{1-\epsilon}{\beta} - q(u_{k_0} - z_{k_0}) > \frac{1-\epsilon}{\beta} - \frac{\frac{1}{2}+\epsilon}{\beta} = \frac{\frac{1}{2}-2\epsilon}{\beta} > 0.$$

So, $\left|u_{k_0}(1) + u_{k_0}(j) + u_{k_0}(j+1)\right| > 0$. Similarly, $\left|u_{k_0+2}(1) + u_{k_0+2}(j) + u_{k_0+2}(j+1)\right| > 0$.

This is a contradiction, since $N_{k_0} + 1 < N_{k_0+1}$.

In a similar way we can get a contradiction to the other case also.

3. Conclusion

In the Theorem 2.7., we prove that $(l^p, |.|_\beta), 1 < p < \infty,$ has the fpp for all $\beta > 0$ if the seminorm q satisfies certain condition but in general the question whether these spaces have the fpp or not for all seminorm q remains open.

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